A new bias correction technique for Weibull parametric estimation

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Abstract

Weibull statistics is a key tool in quality assessment of mechanical product life. This article presents a new technique for computing the inherent bias from a Weibull parametric estimation using maximum likelihood (such a bias being non-negligible for tests with small sample size). This technique extends the range of censoring schemes on which the bias computation is accurate. It is based on Monte Carlo simulations of data following the same scheme as the experimental data. This method applies for Weibull 2-parameters data as well as for a newly introduced distribution, the Advanced Weibull, which combines properties from the Weibull 2-parameters and 3-parameters distributions.

Introduction

To model the randomness of some physical phenomenon like fatigue of material or mechanical product life, the Weibull statistical distribution is often used. It has been introduced in the setting of material strength by Waloddi Weibull ([11]) and later was extended to a wide range of types of experimental data ([12]). Today, the Weibull distribution is extensively used together with its special case, the exponential distribution. For a general overview of this distribution, see [3]. The Weibull distribution itself possesses two main forms, one with 2 parameters and one with 3 parameters. For clarity, these two distributions are defined using L for the random variable (typically in life duration modeling). They are given with their two most common formulations, the more mathematical form with η as the scale parameter, and a form useful to engineers with a percentile requirement of L. As an example the latter form is given using the 10^{th} percentile, L_{10}, as the scale parameter. This 10^{th} percentile will be kept as scale factor along this article. The shape factor is denoted β and, for the Weibull 3-parameters distribution, the 0 percentile or minimum life is denoted L_0 (guaranteed life that is 100% certain to be reached). The link between the two forms is given by $\eta = \frac{L_{10}-L_0}{(-\ln(0.9))^{1/\beta}}$, with $L_0 = 0$ for the Weibull 2-parameters distribution.

- Weibull 2-parameters distribution:

$$\mathbb{P}(L > x) = \exp\left(-\left(\frac{x}{\eta}\right)^{\beta}\right) = 0.9^{\left(\frac{x}{L_{10}}\right)^{\beta}} \quad \text{with } \eta, \beta, L_{10} > 0\,;$$

- Weibull 3-parameters distribution:

$$\mathbb{P}(L > x) = \exp\left(-\left(\frac{x-L_0}{\eta}\right)^{\beta}\right) = 0.9^{\left(\frac{x-L_0}{L_{10}-L_0}\right)^{\beta}} \quad \text{with } \eta, \beta, L_{10} > 0, L_0 \geq 0\,.$$

The Weibull 2-parameters is widely used (especially for life duration of mechanical components leading to key quality assessment) since it keeps a good balance between the adaptability of the density to a specific phenomenon and the ability to have its parameters estimated. For clarity, the rest of the article uses the terminology of life modeling of mechanical components where failure and suspension stand for exact and censored data. Within this field, life tests are performed, leading to failure and suspension times, from which the Weibull parameters need to be estimated.

For the Weibull 2-parameters distribution, the classical method used for life analysis of mechanical components is the Maximum Likelihood Estimation (MLE). This method is known to be biased (see for instance [6]), this bias being non-negligible for small sample size used in testing, less than 30 items. This bias is illustrated in Figures 1 and 2. Acknowledged median bias correction technique (for the MLE estimation) was developed to obtain accurate estimates together with confidence bounds. The current bias correction method in life analysis of mechanical components uses correction factors computed from Monte Carlo simulations and applied to non-censored data or Type II censored data (where suspended data occurred at the time of last failure). For a complete explanation on this bias correction techniques, see [1], [10] and [6, 7]. See also the recent article ([8]) referring to a software able to proceed such bias correction. The present article is focusing on improving this bias correction technique for test data including general censoring scenarios.

A specific feature from the Weibull 2-parameters distribution is that a failure can occur from the very beginning (the 0 percentile L_0 life is null). Some products might have the property that a non-zero minimum life exists ($L_0 > 0$), for instance when the failure process needs a minimum time to be initiated. In that case, the 3 parameters Weibull distribution fits better the life span. Nevertheless, the bias correction method mentioned above cannot be adapted to the 3 parameters Weibull distribution without the pre-knowledge of the L_0 value. Several methods try to cope with this issue by evaluating the L_0 value that will lead to the best estimation for the two other parameters, for instance using a simplex approach (see [2, 9]). Such methods, up to the knowledge of the author, rely on having a large sample size since the L_0 estimation techniques are similar to a curve fitting which allows no bias correction. In addition a recent study of available Weibull 3-parameters estimation softwares concludes that discrepancies hold between various softwares for identical initial data (see [4]).

The current article is introducing a new distribution based on a 2 parameters Weibull one, with a modification leading to a non-zero minimum threshold (or life). This minimum threshold will be fully determined by the standard two parameters (β, L_{10}) or (β, η) in a way that makes the bias correction techniques suitable for Maximum Likelihood parametric estimations.

Bias correction factors for more general censoring

The purpose of the bias correction (for Maximum Likelihood Estimation) method is to compute, from Monte Carlo simulations, correction factors that will be applied to the raw estimates obtained by the Maximum Likelihood to correct the inherent bias. To reach this aim, the Monte Carlo simulations need to be run for reference values for the parameters ($L_{10} = 1$ and $\beta = 1$). Then, the theory ([1], [10] and [6, 7]) ensures that these correction factors are independent from the true value of the parameters and can then be used on real data where, by definition, the true values of the parameters are unknown.

In addition, the Monte Carlo simulations need to generate test data that follows the same scenario (in terms of failures and suspensions) as in the real case for which this bias correction factors will be used. This is the reason why the traditional bias correction techniques is fully accurate for uncensored or type-II censored data (because such scenarios can be perfectly simulated beforehand since they depend only on the sample size and the number of failures). This is clearly shown in Figures 1 and 2 where $10,000$ tests were simulated, all of them analyzed with and without bias correction in order to get point estimates of the L_{10} value (set at 1). The accuracy can then be evaluated while sorting the L_{10} estimates and check that the resulting curve in well centered at 1. This is the fact with bias correction, while using the raw estimates, the L_{10} is overestimated by more than 10%. This error depends strongly on the number of tested items, but the choice made for Figures 1 and 2 (15 to 20 items) is typical for industries of mechanical products like bearings.

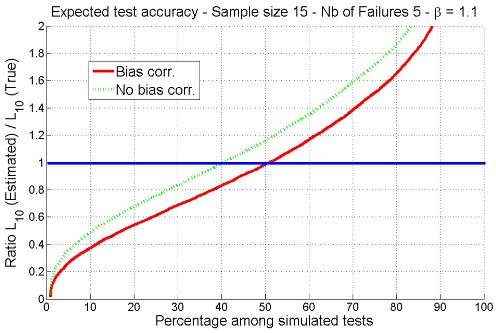

Figure 1: Test accuracy on L_{10} with or without Bias correction

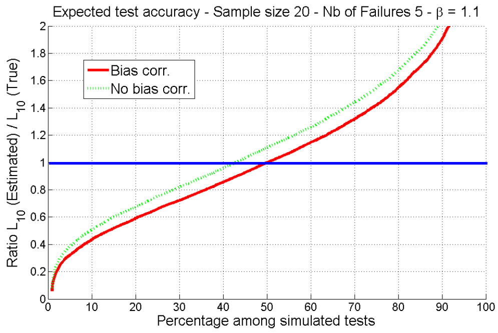

Figure 2: Test accuracy on L_{10} with or without Bias correction

In practice, this method is applied on a wider range of test data and then a remaining bias is kept when the test data deviates from the uncensored or type II censored cases. This is the case with Type-I censoring or when suspensions occur at early stage (in between the failures) or later than the last failure. This point is illustrated in Figure 11 where the former bias correction technique is underestimating the value of β.

The proposed new methodology for computing the bias correction factors manages to compensate this weakness in scenarios alternating uncensored and right censored data. Nowadays, it is doable time wise to perform the full Monte Carlo simulations needed for the bias correction factors for any single data set. Therefore, it is possible to modify the simulated data to fit the exact scenario of the test data from which the Weibull parameters will be estimated.

Note that the new methodology will lead to a computation time of about 3 minutes (CPU time) on a single laptop for a sample size of 30 items, using $1,000,000$ runs to compute the bias correction factors. The latter amount of runs ensures consistency in the bias correction computation.

Monte Carlo simulations

Pre-computation of the bias correction factors is possible for some specific test scenarios. In particular, it is straightforward for uncensored and Type-II censored tests since the only information needed are the sample size N and the number of failures M ($M \leq N$). A matrix of bias correction factors can then be created and used for further test data. The purpose of this section is to show a way to generate random data according to a general scenario defined only by the sequence of failures and suspensions and their relative ratios on a logarithmic scale. To apply the same methodology to more general test scenario, all the simulated tests need to follow the dedicated test scenarios.

First, some notations from the test data are needed:

$$\begin{aligned}
(F_i)_{i=1}^{M} &= \text{recorded failure times} \\
K_0 &= \text{number of suspensions within } (0, F_1) \\
(K_i)_{i=1}^{M-1} &= \text{number of suspensions within } [F_i, F_{i+1}) \\
K_M &= \text{number of suspensions within } [F_M, +\infty) \\
(S_j^0)_{j=1}^{K_0} &= \text{recorded suspension times within } (0, F_1) \text{ (only if } K_0 > 0) \\
(S_j^i)_{j=1}^{K_i} &= \text{recorded suspension times within } [F_i, F_{i+1}) \text{ (only if } K_i > 0) \\
(S_j^M)_{j=1}^{K_M} &= \text{recorded suspension times within } [F_M, +\infty) \text{ (only if } K_M > 0)
\end{aligned}$$

For the purpose of the Monte Carlo simulation, N random numbers are generated according to the desired distribution (Weibull 2-parameters) representing the life times of N items. Among them, some will be kept as failure times and some will be modified and turned into suspension times. The aim is to get at the end a sequence of failure and suspension times following the same scenario as the test data. Here, a scenario means the alternate sequence of failures and suspensions together with criterion on ratios between suspension times and their closest failure time(s). See the different steps hereafter for more details.

Similar notations for the simulated data are defined:

$$\begin{aligned}
(f_i)_{i=1}^{M} &= \text{simulated failure times} \\
(s_j^0)_{j=1}^{K_0} &= \text{simulated suspension times within } (0, f_1) \text{ (only if } K_0 > 0) \\
(s_j^i)_{j=1}^{K_i} &= \text{simulated suspension times within } [f_i, f_{i+1}) \text{ (only if } K_i > 0) \\
(s_j^M)_{j=1}^{K_M} &= \text{simulated suspension times within } [f_M, +\infty) \text{ (only if } K_M > 0)
\end{aligned}$$

The way to get the simulated failure and suspension times listed above from the life times of the N items (Weibull 2-parameters randomly generated numbers with reference parameters $L_{10} = 1$ and $\beta = 1$) is explained step by step. Note that, as for the former bias correction method, the choice for the reference values is irrelevant since the bias correction factors are normalized.

Step 0.

- Step 0.a: (only if $K_0 > 0$) Choose uniformly, among the N items, the K_0 ones to be suspended before the first failure.
- Step 0.b: Set f_1 as the minimum time among the $N - K_0$ remaining items
- Step 0.c: (only if $K_0 > 0$) Set (s_j^0) as, $s_j^0 = S_j^0 \times \dfrac{f_1}{F_1}$, so that, $\forall j = 1$ to K_0 , $\dfrac{s_j^0}{f_1} = \dfrac{S_j^0}{F_1}$.

Step 1.

- Step 1.a: (only if $K_1 > 0$) Choose uniformly, among the $N - 1 - K_0$ remaining items, the K_1 ones to be suspended between the first and the second failure.
- Step 1.b: Set f_2 as the minimum time among the $N - 1 - K_0 - K_1$ remaining items
- Step 1.c: (only if $K_1 > 0$) Set (s_j^1) ($\forall j = 1$ to K_1) as

$$\log\left(s_j^1\right) = \frac{\log\left(\dfrac{F_2}{S_j^1}\right)}{\log\left(\dfrac{F_2}{F_1}\right)} \times \log\left(f_1\right) + \frac{\log\left(\dfrac{S_j^1}{F_1}\right)}{\log\left(\dfrac{F_2}{F_1}\right)} \times \log\left(f_2\right) ,$$

so that, for each j, $\log(s_j^1)$ is the center of mass of $\log(f_1)$ and $\log(f_2)$ with the same weights as $\log(S_j^1)$ being the center of mass of $\log(F_1)$ and $\log(F_2)$

Steps 1.a, 1.b and 1.c are repeated $M-2$ times using each time the lower and lower number of remaining items. For clarity, the general i^{th} step is presented ($i = 1$ to $M-1$)

Step i.

- Step $i.a$: (only if $K_i > 0$) Choose uniformly, among the $N - i - (K_0 + K_1 + \cdots + K_{i-1})$ remaining items, the K_i ones to be suspended between the i^{th} and the $(i+1)^{\text{th}}$ failure.
- Step $i.b$: Set f_{i+1} as the minimum time among the $N - i - (K_0 + K_1 + \cdots + K_i)$ remaining items
- Step $i.c$: (only if $K_i > 0$) Set (s_j^i) ($\forall j = 1$ to K_i) as

$$\log\left(s_j^i\right) = \frac{\log\left(\dfrac{F_{i+1}}{S_j^i}\right)}{\log\left(\dfrac{F_{i+1}}{F_i}\right)} \times \log\left(f_i\right) + \frac{\log\left(\dfrac{S_j^i}{F_i}\right)}{\log\left(\dfrac{F_{i+1}}{F_i}\right)} \times \log\left(f_{i+1}\right) ,$$

so that, for each j, $\log(s_j^i)$ is the center of mass of $\log(f_i)$ and $\log(f_{i+1})$ with the same weights as $\log(S_j^i)$ being the center of mass of $\log(F_i)$ and $\log(F_{i+1})$

Step M. The M^{th} step deals with the suspensions occurring after the last failure (if $K_M > 0$). Set (s_j^M) as $s_j^M = S_j^i \times \dfrac{f_M}{F_M}$, so that, $\forall j = 1$ to K_M, $\dfrac{s_j^M}{f_M} = \dfrac{S_j^M}{F_M}$.

At the end of this process, the full string of times is generated following the test data scenario in terms of the order of suspensions, failures and relatives ratios (on a logarithmic scale).

Such a process is done to each of the $1,000,000$ simulated Weibull set of times (Failures and Suspensions). Resulting times are then analyzed as simulated test data via the Maximum Likelihood Estimation to get raw estimates of β and L_{10} (or η) as for the current bias correction factors computation. These new estimates being then compared to the reference values in order to quantify the bias corresponding to the specific scenario

Bias correction factors

For the sake of completeness, the computation of the correction factors is explained hereafter (see [1], [10] and [6, 7]). They are given by the following formula (1) applied to each of the Monte Carlo run (where all the parameters are known):

$$CF(Lq) = \widehat{\beta} \ln\left(\frac{L_q}{\widehat{L_q}}\right) \quad \text{and} \quad CF(\beta) = \frac{\beta}{\widehat{\beta}} \cdot \tag{1}$$

The q^{th} percentile is $L_q = \ln(1 - q/100) / (\ln 0.9)$ for the reference values $\beta = 1$ and $L_{10} = 1$, so that, from (1), the final formulae for the correction factors are

$$CF(Lq) = \widehat{\beta} \ln\left(\frac{\dfrac{\ln(1-q/100)}{\ln 0.9}}{\widehat{L_q}}\right) \quad \text{and} \quad CF(\beta) = \frac{1}{\widehat{\beta}} \cdot \tag{2}$$

Along the full simulation campaign, values for the correction factors are recorded, ranked and several (like the 5^{th}, 10^{th}, 50^{th}, 90^{th} and 95^{th}) percentiles are registered leading to

$$[CF(L_q)_5\ ;\ CF(L_q)_{10}\ ;\ CF(L_q)_{50}\ ;\ CF(L_q)_{90}\ ;\ CF(L_q)_{95}]\ ,$$

$$[CF(\beta)_5\ ;\ CF(\beta)_{10}\ ;\ CF(\beta)_{50}\ ;\ CF(\beta)_{90}\ ;\ CF(\beta)_{95}]\ .$$

This way, after having obtained the Maximum Likelihood estimations $(\widehat{\beta}, \widehat{L_q})$ of parameters from a set of data, median bias corrected estimates can be computed as follows:

$$\widehat{L_{q,50}} = \widehat{L_q} \times \exp\left(\frac{CF(L_q)_{50}}{\widehat{\beta}}\right) \quad \text{and} \quad \widehat{\beta}_{50} = \widehat{\beta} \times CF(\beta)_{50}\,. \tag{3}$$

Similar formulae stand for the other percentiles leading to any desired bias corrected confidence bounds. For example the values for $\widehat{L_{q,5}}$, $\widehat{L_{q,10}}$, $\widehat{L_{q,90}}$ and $\widehat{L_{q,95}}$ together with the $\widehat{\beta}_5$, $\widehat{\beta}_{10}$, $\widehat{\beta}_{90}$ and $\widehat{\beta}_{95}$ are obtained by replacing, in (3), $(\widehat{L_{q50}}, \widehat{\beta}_{50})$ by the ad hoc correction factors $(\widehat{L_{q,5}}, \widehat{\beta}_5)$, $(\widehat{L_{q,10}}, \widehat{\beta}_{10})$, $(\widehat{L_{q,90}}, \widehat{\beta}_{90})$ and $(\widehat{L_{q,95}}, \widehat{\beta}_{95})$.

Computer validation

The two bias correction methods (for Maximum Likelihood Estimations) are compared:

- Bias Correction 1. Original method based on correction factors computed from only Type II scenarios.
- Bias Correction 2. New method based on correction factors for the actual scenario of the studied test data.

To give more details on the improved bias correction techniques, results from Monte Carlo simulations are presented hereafter. The two bias correction methods are by definition perfectly accurate for type II censored data or uncensored data. The robustness is checked for Type I censoring. Then, 2 test scenarios are simulated (all with 30 tested items and $1,000$ runs):

- Test 1. Type I test with suspension time $T = 2 \times L_{10}$
- Test 2. Type I test with suspension time $T = 3 \times L_{10}$

The suspension times must guarantee almost all simulated test to reach enough failures to make the parameters estimation valid. Indeed, when too few failures (2 or less) are obtained, the test has to be discarded since no valuable statistical Weibull estimation can be performed. This discarding should be restricted to a very limited amount of runs. The verification is based on Weibull 2 parameters data. The scale parameter L_{10} is fixed at 1 without losing any generality. The β is chosen at 1.5, 2 and 3.

For each of these tests, accuracy and precision of the estimation are evaluated onto the L_{10}, L_{50} and β. The evaluation of the accuracy means to quantify the potential remaining bias in the estimation. The evaluation of the precision means to quantify the width of the uncertainty of the estimation.

- Accuracy: the accuracy is evaluated via the ratio between the median estimate (of L_{10}, L_{50} and β) and the true value.
- Precision: the precision is evaluated via the ratio between the upper and lower bounds of the 90% confidence intervals of L_{10}, L_{50} and β, the first two ratios being put at the power β: $R(L_{10})^\beta = (Ratio\,L_{10})^\beta$; $R(L_{50})^\beta = (Ratio\,L_{50})^\beta$; $R(\beta) = (Ratio\,\beta)$.

The ratios $R(L_{10})$ and $R(L_{50})$ are strongly dependent on β. Indeed, low β leads naturally to wider confidence intervals since the failures are more spread out. Therefore, those ratios are put to the power β to get a quantity that is less dependent on β. Indeed, for Type II tests, such ratios to the power β are β independent.

Precision ratios are computed for each simulated test. Then a comparison can be drawn from their distributions, through the median and relevant percentiles. For the accuracy evaluation, the ratios are computed using a transformation to get only ratios larger than 1, taking the exponential of the absolute value of the logarithm of each ratio (see Table 5). Thus, estimations leading to half or twice the true value give the same ratio.

The complete results are presented in appendix (Table 2 to 13) where all quantities are given through their 50^{th} and 90^{th} percentiles. The exponent (1) or (2) in Table 2 to 13 refers respectively to the standard bias correction and the new one proposed in this article. The purpose of these tables is to compare the performance (accuracy and precision) of these two bias correction techniques. Even if results are similar for high β and high suspension time, all outputs are better with the new bias correction techniques. Moreover, significant difference is observed for low β.

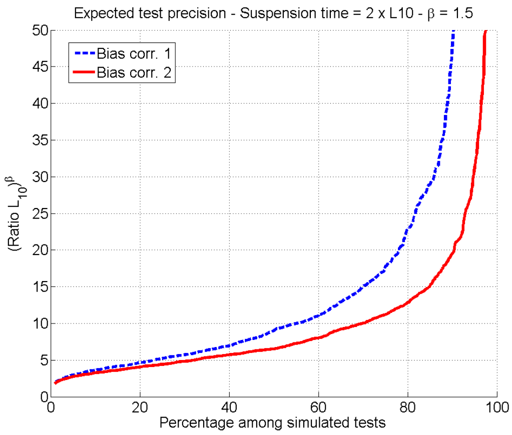

Figure 3: Test precision on L_{10}

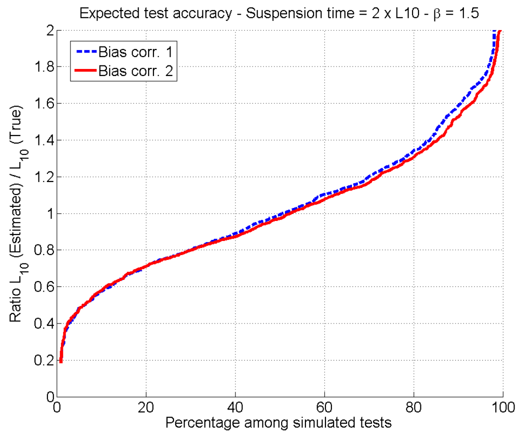

Figure 4: Test accuracy on L_{10}

In addition Figure 3 to 8 are showing the improved precision and accuracy of the new bias correction techniques in an illustrative case, where $\beta = 1.5$ and the suspension time is $T = 2 \times L_{10}$. Indeed, the precision ratios $R(L_{10})^{\beta}$, $R(L_{50})^{\beta}$ and $R(\beta)$ are all three closer to 1 with the new techniques (see Figures 3, 5 and 7). In particular, the new bias correction techniques prevents from extreme uncertainty onto some parameter estimation as it can be observed in the high percentiles for the test precision (at 80%, $R(L_{10})^{\beta}$ goes from 23 to 13 and $R(L_{50})^{\beta}$ goes from 50 to 15). This is of importance since an estimation of L_{10}, for instance, with a ratio 50 between the upper and lower bound of the confidence interval leads to unacceptable results. In such case the results from the test cannot be used. From Figure 3 this event occurs with a 10% risk when using the traditional bias correction techniques, but only with a 3% risk when using the new bias correction techniques.

In Figure 7, the steps observed for the precision ratio are due to the fact that the confidence bounds are computed from the median estimation of β via a multiplicative factor that depends only on the number of failures (the total sample size being constant and equal to 30). Therefore, the several plateau correspond to identical number of failures. Figures 6 and 8 give evidence that the classical bias correction techniques lead to a global overestimation of the L_{50} and underestimation of the β. As for the accuracy onto the L_{10}, both techniques are equivalent with a light premium to the new one (Figure 4).

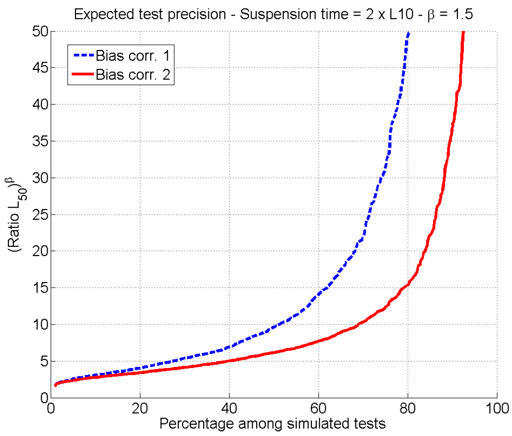

Figure 5: Test precision on L_{50}

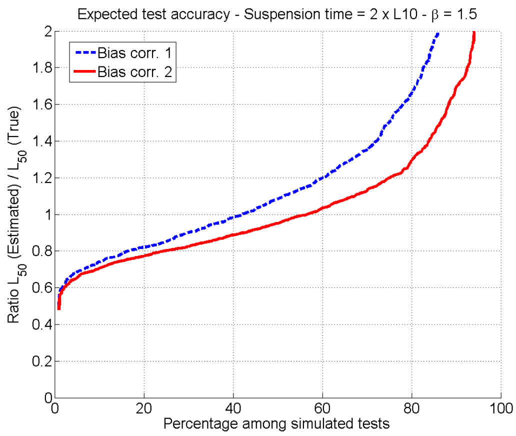

Figure 6: Test accuracy on L_{50}

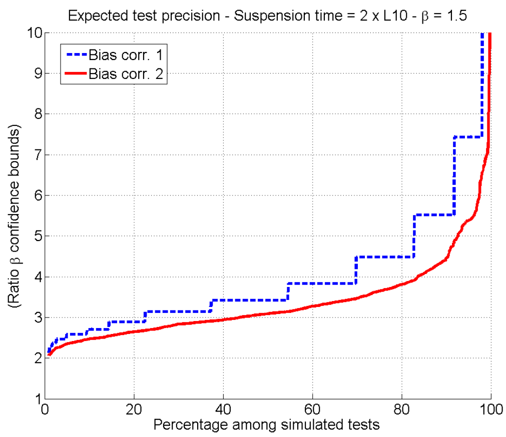

Figure 7: Test precision on β

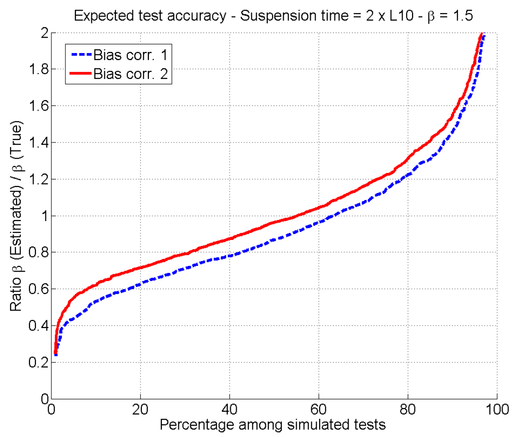

Figure 8: Test accuracy on β

In addition to the computer validation, three real examples of test data (normalized) are shown to illustrate the gain in accuracy and precision on concrete data that are deviating from the Type II scenario (see Figures 9 to 11). The test data have the following settings: Data 1 (30 items and 11 Failures), Data 2 (27 items and 6 Failures) and Data 3 (30 items and 4 Failures). Data 1 is a Type I test (all suspensions at the same time). Data 3 is an extreme case since the test encountered numerous suspensions much later than the last failure, ending in a severe underestimation of the β parameters which has drastic effect onto the accuracy and the precision. Data 2 encounters late suspensions closer to the final failures, but it contains several suspensions in the course of the tests so earlier than the last failure.

Note (from Table 1) that the differences observed on the Data 1 to 3 (see Figures 9 to 11) are larger than the one shown in the simulations (Figure 3 to 8) because the scenarios from the real test data are deviating from the Type II one to a larger extend.

Both computer validation and real examples are enlightening the significant benefit of the new methodology for bias correction.

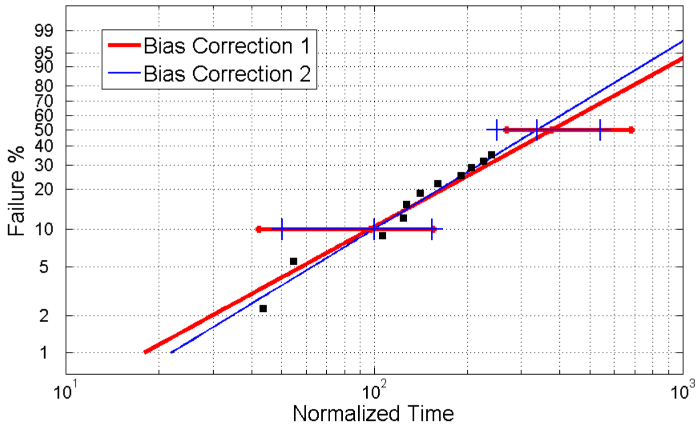

Figure 9: Bias correction comparison (Data 1)

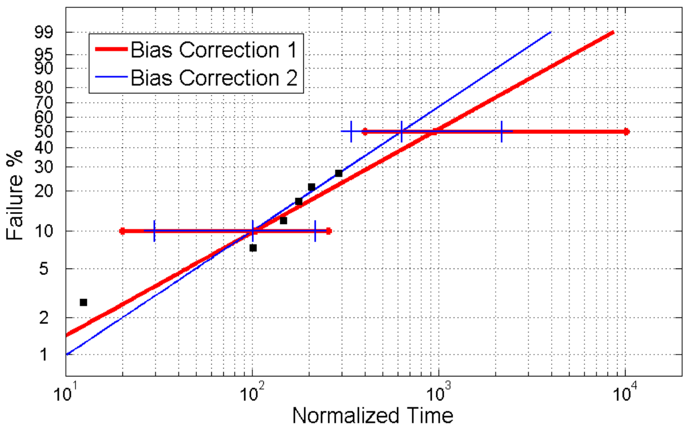

Figure 10: Bias correction comparison (Data 2)

Table 1: Precision and Accuracy deviation on 3 examples

	Acc. L_{10}	Acc. L_{50}	Acc. β	Prec. L_{10}	Prec. L_{50}	Prec. β
Data 1	29%	83 %	86 %	89 %	99 %	64 % %
Data 2	3 %	33 %	20 %	42 %	75 %	31 %
Data 3	3%	10%	11%	17 %	15%	10%

Definition of the Advanced Weibull distribution

Some life models require the use of a minimum life, a quantity that is certain to be reached without failure. This quantity is denoted L_0. The Weibull variant with such a minimum life is the 3-parameters

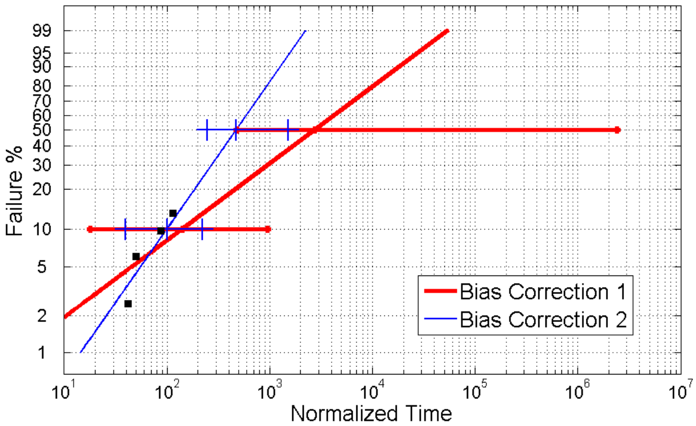

Figure 11: Bias correction comparison (Data 3)

distribution defined in the introduction. When it comes to the parametric estimation, the Weibull 3-parameters is not compatible with the bias correction technique, unless the minimum life L_0 is fixed. In practice, this L_0 is estimated prior to the other parameters (β and L_{10}). In addition, no confidence interval is given. The current section proposes a new statistical distribution able to capture a minimum life and to keep the bias correction technique valid, even on the L_0 itself.

Definition

The definition of the Advanced Weibull distribution is:

$$\mathbb{P}(L > x) = 0.9^{\left(\frac{1}{1-\alpha^{\beta_0}}\left(\left(\frac{x}{L_{10}}\right)^{\beta} - \alpha^{\beta_0}\right)\right)} \quad \text{with} \quad \beta_0, \beta, L_{10} > 0 \text{ and } 1 > \alpha \geq 0\,, \tag{4}$$

where β and L_{10} are parameters and β_0 and α are structural constants.

The idea behind this new distribution is to set a constant α ($0 \leq \alpha < 1$) representing the ratio between the minimum life L_0 and the L_{10} life at a chosen slope parameter β_0. As an example, for rolling bearings, the choice for α can typically be 0.05, approximating the ISO standard [5] ratio $L_{0.05}/L_{10} = 0.05$. For β_0, the choice can also follow the ISO standard [5] being $\beta_0 = 1.5$.

The choice for β_0 and α has to be made beforehand and depend on the physical process examined (electronic component or rolling bearing for instance) but not on the product size or design. The initial choice is then valid for the entire product range so that β_0 and α are structural constants and not parameters to be estimated. That is why this distribution has only 2 parameters to estimate.

The formula (4) is defined for $x \geq \alpha^{\frac{\beta_0}{\beta}} L_{10}$ and leads to a non-zero minimum life L_0 (meaning $\mathbb{P}(L > L_0) = 1$) whose value is $L_0 = \alpha^{\frac{\beta_0}{\beta}} L_{10}$.

It should be first noticed that the above formula corresponds to a proper statistical distribution (going from 1 to 0 when x goes from L_0 to infinity). Moreover, the L_{10} parameter still corresponds to the 90% reliability, that is $\mathbb{P}(L > L_{10}) = 0.9$. The possibility to state a non-zero minimum life gives the flexibility to have a better fit with real data for high reliability. From a theoretical point of view, this flexibility is obviously weaker than the one of the 3 parameters Weibull where the minimum life L_0 is independent from the 2 other parameters. But its gain in terms of bias correction for the parameters estimation and confidence bounds is an asset that makes this new distribution more advanced. Now, the focus is to prove that this distribution keeps the key characteristic of the classical Weibull 2 parameters distribution, namely the existence of median-bias correction techniques.

Indeed, an easier modification would have been to change L_0 into αL_{10} in the definition of the Weibull 3 parameters distribution in order to obtain a minimum threshold seen as a function of the other parameters. But, with this simplification, the bias correction techniques fails since the correction factors are no longer independent from the reference values chosen for the simulations. This makes the correction factors not useable.

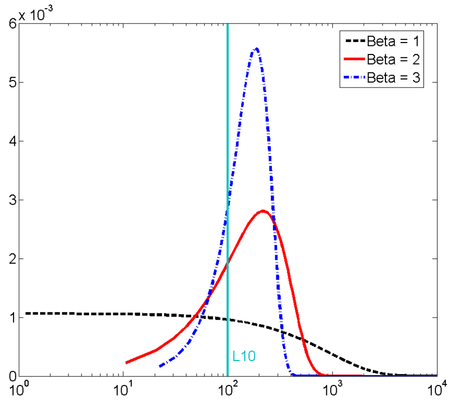

Figure 12: Advanced Weibull density function

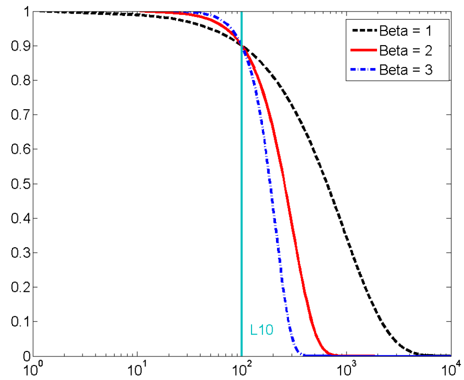

Figure 13: Advanced Weibull cumulative probability

Plots (Figures 12 and 13) are shown, representing the cumulative probability (decreasing) and the density function for $\beta_0 = 1.5$, $\alpha = 0.05$ and $L_{10} = 100$. The value of β taking values 1, 2 and 3. Note that a change in β_0 will almost not affect the density functions (the slight change cannot be seen graphically) but only the value of L_0. The values of α and L_{10} applies as scale factors, their effect being then not shown.

Maximum Log-Likelihood estimation

To prove that a median-bias correction, for the maximum likelihood estimation of the scale L_{10} and the slope β parameters, is possible for the Advanced Weibull distribution, the plan is to follow the method used for the classical Weibull 2 parameters distribution ([1, 6, 7, 10]). Let define

$$C = C(\alpha, \beta_0) = \frac{1}{(1-\alpha^{\beta_0})}\left(-\ln(0.9)\right),$$

such that the cumulative probability of the Advanced Weibull distribution (4) can be written as

$$\mathbb{P}(L > x) = \exp\left(-C\left(\left(\frac{x}{L_{10}}\right)^{\beta} - \alpha^{\beta_0}\right)\right). \tag{5}$$

The probability density (derivative of the cumulative probability) function is

$$f(x) = -\frac{\partial \mathbb{P}(L > x)}{\partial x} = C\left(\frac{\beta}{L_{10}}\left(\frac{x}{L_{10}}\right)^{\beta-1}\right) \times \exp\left(-C\left(\left(\frac{x}{L_{10}}\right)^{\beta} - \alpha^{\beta_0}\right)\right). \tag{6}$$

Estimating the distribution parameters requires data from either tests or field. For life duration, this data consists of failure and suspension times. The principle of the Maximum Likelihood method is to find the parameters that render this data the most probable (highest likelihood) to occur. The Log-Likelihood function Λ for n items with r failure times $(x_i)_{i=1}^{r}$ and $(n-r)$ suspension times $(x_i)_{i=r+1}^{n}$ is by definition

$$\Lambda = \underbrace{\sum_{i=1}^{r} \ln(f(x_i))}_{\Lambda_1} + \underbrace{\sum_{i=r+1}^{n} \ln(\mathbb{P}(L > x_i))}_{\Lambda_2}, \tag{7}$$

where the two terms on the right hand side can be rewritten using (5) and (6):

$$\Lambda_1 = r\ln(C) + r\ln\left(\frac{\beta}{L_{10}}\right) + \sum_{i=1}^{r} \ln\left(\frac{x_i}{L_{10}}\right)^{\beta-1} - C\sum_{i=1}^{r}\left(\frac{x_i}{L_{10}}\right)^{\beta} + C\alpha^{\beta_0} r \tag{8}$$

and

$$\Lambda_2 = -C\sum_{i=r+1}^{n}\left(\frac{x_i}{L_{10}}\right)^{\beta} + C\alpha^{\beta_0}(n-r)\,. \tag{9}$$

Combining the two terms (8) and (9) into (7), the log-Likelihood function Λ becomes

$$\Lambda = r\ln(C) + r\ln(\beta) - r\ln(L_{10}) + (\beta-1)\sum_{i=1}^{r}\ln\left(\frac{x_i}{L_{10}}\right) - C\sum_{i=1}^{n}\left(\frac{x_i}{L_{10}}\right)^{\beta} + C\alpha^{\beta_0} n\,.$$

Now, the derivative of Λ with respect to β and L_{10} gives

$$\frac{\partial \Lambda}{\partial \beta}(\beta, L_{10}) = \frac{r}{\beta} + \sum_{i=1}^{r}\ln\left(\frac{x_i}{L_{10}}\right) - C\sum_{i=1}^{n}\left(\frac{x_i}{L_{10}}\right)^{\beta}\ln\left(\frac{x_i}{L_{10}}\right)$$

and

$$\frac{\partial \Lambda}{\partial L_{10}}(\beta, L_{10}) = -\frac{\beta r}{L_{10}} + \frac{C\beta}{L_{10}}\sum_{i=1}^{n}\left(\frac{x_i}{L_{10}}\right)^{\beta}.$$

The next step is to solve the following system of equations $(\mathcal{S})$ whose solutions are the raw estimates $\left(\widehat{\beta}, \widehat{L_{10}}\right)$ for the unknown parameters (β, L_{10}):

$$(\mathcal{S}) = \left(\frac{\partial \Lambda}{\partial \beta}\left(\widehat{\beta}, \widehat{L_{10}}\right) = 0\,;\ \frac{\partial \Lambda}{\partial L_{10}}\left(\widehat{\beta}, \widehat{L_{10}}\right) = 0\right)$$

The system can be written as

$$(\mathcal{S}) = \left(\begin{array}{c} \dfrac{1}{\widehat{\beta}} + \dfrac{\sum_{i=1}^{r}\ln(x_i)}{r} - \ln\left(\widehat{L_{10}}\right) - \dfrac{C\sum_{i=1}^{n} x_i^{\widehat{\beta}}\ln(x_i)}{r\widehat{L_{10}}^{\widehat{\beta}}} + \dfrac{C\ln\left(\widehat{L_{10}}\right)\sum_{i=1}^{n} x_i^{\widehat{\beta}}}{r\widehat{L_{10}}^{\widehat{\beta}}} = 0\,; \\ r = \dfrac{C}{\widehat{L_{10}}^{\widehat{\beta}}} \times \sum_{i=1}^{n} x_i^{\widehat{\beta}} \end{array}\right) \tag{10}$$

So, by replacing r by its expression coming from the second equation in (10), into the last two terms of the first equation of (10), the system becomes

$$(\mathcal{S}) = \left(\frac{1}{\widehat{\beta}} + \frac{\sum_{i=1}^{r} \ln(x_i)}{r} - \frac{\sum_{i=1}^{n} x_i^{\widehat{\beta}} \ln(x_i)}{\sum_{i=1}^{n} x_i^{\widehat{\beta}}} = 0\,;\ \widehat{L_{10}} = \left(\frac{C \sum_{i=1}^{n} x_i^{\widehat{\beta}}}{r} \right)^{\frac{1}{\widehat{\beta}}} \right) \tag{11}$$

At this point, the same steps as for the classical Weibull 2 parameters distribution can be followed to construct the median bias correction techniques, namely to show the remarkable result that bias computed onto a reference distribution with known parameters ($\beta = 1$ and $L_{10} = 1$ typically) can be applied for the estimation of unknown parameters in a way that is independent of the parameter values. Nevertheless, it is simpler to compare the system $(\mathcal{S})$ in (11) with the one for the classical Weibull 2 parameters distribution, called $(\mathcal{S}(Clas.))$ (defined in (12)), whose solutions are denoted $\left(\widehat{\beta}(Clas.), \widehat{L_{10}}(Clas.)\right)$. From this comparison, the estimates for the Advanced Weibull distribution can be deducted from the estimates for the classical Weibull 2-parameters distribution as explained hereafter. For completeness, the system $(\mathcal{S}(Clas.))$ is defined:

$$(\mathcal{S}(Clas.)) = \left(\begin{array}{c} \dfrac{1}{\widehat{\beta}(Clas.)} + \dfrac{\sum_{i=1}^{r} \ln(x_i)}{r} - \dfrac{\sum_{i=1}^{n} x_i^{\widehat{\beta}(Clas.)} \ln(x_i)}{\sum_{i=1}^{n} x_i^{\widehat{\beta}(Clas.)}} = 0\,; \\ \widehat{L_{10}}(Clas.) = \left(\dfrac{1}{r}(-\ln 0.9) \sum_{i=1}^{n} x_i^{\widehat{\beta}(Clas.)} \right)^{\frac{1}{\widehat{\beta}(Clas.)}} \end{array} \right) \tag{12}$$

Namely, comparing the two systems $(\mathcal{S})$ and $(\mathcal{S}(Clas.))$ for identical set $(x_i)_{i=1}^{n}$, the link between the solutions leads to

$$\widehat{\beta} = \widehat{\beta}(Clas.) \ \text{ and } \ \widehat{L_{10}} = \left(1 - \alpha^{\beta_0}\right)^{\frac{-1}{\widehat{\beta}(Clas.)}} \times \widehat{L_{10}}(Clas.)\,. \tag{13}$$

More generally,

$$\widehat{L_q} = \widehat{\eta}(Clas.) \times \left(-\ln\left(1 - \frac{q}{100}\right) - \ln(0.9) \frac{\alpha^{\beta_0}}{1 - \alpha^{\beta_0}} \right)^{\frac{-1}{\widehat{\beta}(Clas.)}} . \tag{14}$$

Therefore the estimation for the Advanced Weibull distribution can be derived from the solutions of the system of equations for the classical Weibull 2 parameters distribution by the simple modifications (13). This is of importance because it allows using only a standard software (like Matlab) giving the numerical solutions of the system $(\mathcal{S}(Clas.))$.

Correction factors

Similarly to the Weibull 2-parameters, the correction factors are computed from the q^{th} life percentile for the reference values $\beta = 1$ and $L_{10} = 1$:

$$L_q = L_{10} \times \left(\frac{\left(1 - \alpha^{\beta_0}\right) \times \ln(1 - q/100)}{\ln 0.9} + \alpha^{\beta_0} \right)^{\frac{1}{\beta}} = \frac{\left(1 - \alpha^{\beta_0}\right) \times \ln(1 - q/100)}{\ln 0.9} + \alpha^{\beta_0}\,. \tag{15}$$

Replacing (15) into (1), the final formulae for the correction factors are then

$$CF(Lq) = \widehat{\beta} \ln \left(\frac{\frac{\left(1-\alpha^{\beta_0}\right) \times \ln\left(1-q/100\right)}{\ln 0.9} + \alpha^{\beta_0}}{\widehat{L_q}} \right) \quad \text{and} \quad CF(\beta) = \frac{1}{\widehat{\beta}} \cdot \tag{16}$$

These factors need then to be used as in (3) in order to get the bias corrected estimations for the Advanced Weibull distribution.

Discussion

Likewise the Weibull 2-parameters distribution, the Advanced Weibull one benefits from having bias correction techniques for the Maximum Likelihood Estimation of its parameters. This property makes it useful when a non-zero minimum life is needed and when insufficient test data (typically less than 100) are available. When extended test data are available, it is then possible to pick safely a L_0 value by curve fitting in order to pursue with a Weibull 3-parameters which boils down to a shifted Weibull 2-parameters. Nevertheless, with fewer test data, fixing the value of L_0 is not reliable since it does not come with confidence bounds, which are crucial to estimate the precision of any estimation.

The drawback of the Advanced Weibull distribution is that it relies on fixed constants (α,β_0) assumed to be general constants fixed by pre-knowledge on a larger class of items that the specific tested ones. This is typically the case when dealing with bearings for which the ISO standard can provide such constants.

Conclusion

For the Maximum Likelihood Estimation of the Weibull 2-parameters, a new bias correction techniques is introduced with evidence (through examples and Monte Carlo simulations) of its improved accuracy and precision onto the parameter estimation.

An Advanced Weibull statistical distribution is introduced, that allows analyzing test data assuming a non-zero minimum threshold while keeping the use of existing bias correction techniques. This Advanced distribution is typically useful to estimate percentiles at a higher reliability level than 90% together with confidence bounds.

Finally note that the present methodology can readily adapt to sudden death tests following the exact same methodology as for the classical Weibull 2-parameters distribution.

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References

[1] Bain, L. J. and Antle, C. E. (1967). Estimation of parameters in the weibull distribution. *Technometrics*, 9(4):621–627.

[2] Cousineau, D., Brown, S., and Heathcote, A. (2004). Fitting distributions using maximum likelihood: Methods and packages. *Behavior Research Methods, Instruments, & Computers*, 36:742–756.

[3] Dodson, B. (2006). *The Weibull analysis handbook. Milwaukee, Wis: ASQ Quality Press.*

[4] Harper, W., Eschenbach, T., and James, T. (2011). Concerns about maximum likehood estimation for the three-parameter weibull distribution: case study of statistical software. *The American Statistician*, 65(1):44–54.

[5] ISO281 (2007). *International Standard: Rolling Bearings, Dynamic Load Ratings and Rating Life.* 2nd Ed.

[6] Mc Cool, J. (1970). Evaluating weibull endurance data by the method of maximum likelihood. *ASLE Transactions*, 13(3):189–202.

[7] Mc Cool, J. (1986). Using weibull regression to estimate the load-life relationship for rolling bearings. *ASLE Transactions*, 29:91–101.

[8] Mc Cool, J. (2011). Software for weibull inference. *Quality Engineering*, 23:253–264.

[9] Nelder, J. and Mead, R. (1965). A simplex method for function minimization. *The Computer J.*, 7:308–313.

[10] Thoman, D. R., Bain, L. J., and Antle, C. E. (1969). Inferences on the parameters of the weibull distribution. *Technometrics*, 11(3):445–460.

[11] Weibull, W. (1939). Statistical theory of the strength of materials. *Ing. Vetenskaps Akad. Handl.*, 151:1–47.

[12] Weibull, W. (1951). A statistical distribution function of wide applicability. *J. Appl. Mech.*, 18(3):293–297.

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Tables on results and errors

The exponent (1) or (2) in Table 2 to 13 refers respectively to the current bias correction on Maximum Likelihood estimates and the new one proposed in this article.

Table 2: Precision results - $R(L_{10})^\beta$ - Suspension time = $2 \times L_{10}$

$R(L_{10})^\beta$	$\beta=1.5^{(1)}$	$\beta=1.5^{(2)}$	$\beta=2^{(1)}$	$\beta=2^{(2)}$	$\beta=3^{(1)}$	$\beta=3^{(2)}$
50^{th} prc.	9.21	6.59	7.39	6.34	5.83	5.53
90^{th} prc.	48.83	19.62	20.43	15.5	10.59	9.66

Table 3: Precision results - $R(L_{50})^{\beta}$ - Suspension time $= 2 \times L_{10}$

$R(L_{50})^{\beta}$	$\beta\!=\!1.5^{(1)}$	$\beta\!=\!1.5^{(2)}$	$\beta\!=\!2^{(1)}$	$\beta\!=\!2^{(2)}$	$\beta\!=\!3^{(1)}$	$\beta\!=\!3^{(2)}$
50^{th} prc.	9.77	6.2	4.46	3.94	2.51	2.45
90^{th} prc.	434.31	37.33	23.63	13.04	3.73	3.55

Table 4: Precision results - $R(\beta)$ - Suspension time $= 2 \times L_{10}$

$R(\beta)$	$\beta\!=\!1.5^{(1)}$	$\beta\!=\!1.5^{(2)}$	$\beta\!=\!2^{(1)}$	$\beta\!=\!2^{(2)}$	$\beta\!=\!3^{(1)}$	$\beta\!=\!3^{(2)}$
50^{th} prc.	3.43	3.1	2.9	2.7	2.14	2.11
90^{th} prc.	5.52	4.47	3.84	3.5	2.38	2.36

Table 5: Accuracy results - Ratio $(L_{10})est./(L_{10})true$ - Suspension time $= 2 \times L_{10}$

$\exp\left\|\log\left(\frac{L_{10}\,est.}{L_{10}\,true}\right)\right\|$	$\beta\!=\!1.5^{(1)}$	$\beta\!=\!1.5^{(2)}$	$\beta\!=\!2^{(1)}$	$\beta\!=\!2^{(2)}$	$\beta\!=\!3^{(1)}$	$\beta\!=\!3^{(2)}$
50^{th} prc.	1.29	1.28	1.2	1.19	1.12	1.11
90^{th} prc.	1.86	1.8	1.6	1.57	1.33	1.32

Table 6: Accuracy results - Ratio $(L_{50})est./(L_{50})true$ - Suspension time $= 2 \times L_{10}$

$\exp\left\|\log\left(\frac{L_{50}\,est.}{L_{50}\,true}\right)\right\|$	$\beta\!=\!1.5^{(1)}$	$\beta\!=\!1.5^{(2)}$	$\beta\!=\!2^{(1)}$	$\beta\!=\!2^{(2)}$	$\beta\!=\!3^{(1)}$	$\beta\!=\!3^{(2)}$
50^{th} prc.	1.28	1.23	1.15	1.14	1.06	1.06
90^{th} prc.	2.53	1.73	1.55	1.4	1.15	1.15

Table 7: Accuracy results - Ratio $(\beta)est./(\beta)true$ - Suspension time $= 2 \times L_{10}$

$\exp\left\|\log\left(\frac{\beta\,est.}{\beta\,true}\right)\right\|$	$\beta\!=\!1.5^{(1)}$	$\beta\!=\!1.5^{(2)}$	$\beta\!=\!2^{(1)}$	$\beta\!=\!2^{(2)}$	$\beta\!=\!3^{(1)}$	$\beta\!=\!3^{(2)}$
50^{th} prc.	1.32	1.28	1.27	1.24	1.18	1.17
90^{th} prc.	2.04	1.8	1.75	1.67	1.49	1.47

Table 8: Precision results - $R(L_{10})^{\beta}$ - Suspension time $= 3 \times L_{10}$

$R(L_{10})^{\beta}$	$\beta\!=\!1.5^{(1)}$	$\beta\!=\!1.5^{(2)}$	$\beta\!=\!2^{(1)}$	$\beta\!=\!2^{(2)}$	$\beta\!=\!3^{(1)}$	$\beta\!=\!3^{(2)}$
50^{th} prc.	6.63	5.65	5.58	5.24	4.5	4.38
90^{th} prc.	15.64	11.78	10.08	9	6.26	6.04

Table 9: Precision results - $R(L_{50})^{\beta}$ - Suspension time = $3 \times L_{10}$

$R(L_{50})^{\beta}$	$\beta=1.5^{(1)}$	$\beta=1.5^{(2)}$	$\beta=2^{(1)}$	$\beta=2^{(2)}$	$\beta=3^{(1)}$	$\beta=3^{(2)}$
50^{th} prc.	3.36	3.02	2.39	2.3	2.09	2.07
90^{th} prc.	8.58	6	3.43	3.24	2.44	2.42

Table 10: Precision results - $R(\beta)$ - Suspension time = $3 \times L_{10}$

$R(\beta)$	$\beta=1.5^{(1)}$	$\beta=1.5^{(2)}$	$\beta=2^{(1)}$	$\beta=2^{(2)}$	$\beta=3^{(1)}$	$\beta=3^{(2)}$
50^{th} prc.	2.59	2.39	2.09	2.03	1.69	1.67
90^{th} prc.	3.15	2.85	2.29	2.24	1.72	1.72

Table 11: Accuracy results - Ratio $(L_{10})est./(L_{10})true$ - Suspension time = $3 \times L_{10}$

$\exp\left[\log\left(\frac{L_{10}\,est.}{L_{10}\,true}\right)\right]$	$\beta=1.5^{(1)}$	$\beta=1.5^{(2)}$	$\beta=2^{(1)}$	$\beta=2^{(2)}$	$\beta=3^{(1)}$	$\beta=3^{(2)}$
50^{th} prc.	1.24	1.23	1.18	1.18	1.11	1.11
90^{th} prc.	1.78	1.75	1.5	1.48	1.27	1.27

Table 12: Accuracy results - Ratio $(L_{50})est./(L_{50})true$ - Suspension time = $3 \times L_{10}$

$\exp\left[\log\left(\frac{L_{50}\,est.}{L_{50}\,true}\right)\right]$	$\beta=1.5^{(1)}$	$\beta=1.5^{(2)}$	$\beta=2^{(1)}$	$\beta=2^{(2)}$	$\beta=3^{(1)}$	$\beta=3^{(2)}$
50^{th} prc.	1.16	1.14	1.1	1.09	1.05	1.05
90^{th} prc.	1.53	1.41	1.23	1.22	1.13	1.13

Table 13: Accuracy results - Ratio $(\beta)est./(\beta)true$ - Suspension time = $3 \times L_{10}$

$\exp\left[\log\left(\frac{\beta\,est.}{\beta\,true}\right)\right]$	$\beta=1.5^{(1)}$	$\beta=1.5^{(2)}$	$\beta=2^{(1)}$	$\beta=2^{(2)}$	$\beta=3^{(1)}$	$\beta=3^{(2)}$
50^{th} prc.	1.22	1.2	1.16	1.16	1.11	1.11
90^{th} prc.	1.66	1.6	1.46	1.44	1.29	1.29